Year 12 Mathematics Specialist 2019

Test Number 1: Complex Numbers

Resource Free

Name: _______SOLUTIONS___________________________ **Teacher: Mrs Da Cruz**

Marks: **20**

Time Allowed: **20 minutes**

<u>Instructions:</u> You **ARE NOT** permitted any notes or calculator. Show your working where appropriate remembering you must show working for questions worth more than 2 marks.

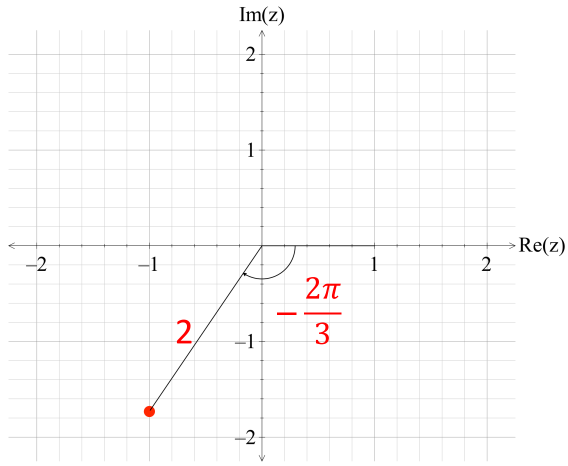
Question 1 **[2 marks]**

Express $z = -1 - \sqrt{3}i$ in polar form.

$$|z| = \sqrt{(-1)^2 + (-3)^2} = 2$$

$$\tan\theta = \frac{-\sqrt{3}}{-1} = \sqrt{3} \quad \Rightarrow \theta = -\frac{2\pi}{3} \qquad (Quadrant\ 3)$$

$$\therefore z = 2cis\left(-\frac{2\pi}{3}\right)$$



Question 2 **[1, 4 = 5 marks]**

Consider $f(z) = 2z^3 - 5z^2 + 22z - 10,\ z \in \mathbb{C}$.

a) Show that $f(0.5) = 0$.

$$f\left(\frac{1}{2}\right) = 2\left(\frac{1}{8}\right) - 5\left(\frac{1}{4}\right) + 22\left(\frac{1}{2}\right) - 10$$
$$= \frac{1}{4} - \frac{5}{4} + 11 - 10$$
$$= -1 + 1$$
$$= 0$$

Must show ½ being substituted in and terms evaluated to second row then zero as a minimum.

b) Find all values, real and complex, for which $f(z) = 0$.

$$\frac{2z^3 - 5z^2 + 22z - 10}{z - \frac{1}{2}} = 2z^2 - 4z + 20$$

$$z^2 - 2z + 10 = 0$$
$$(z-1)^2 - 1 + 10 = 0$$

$$(z-1)^2 = -9$$
$$z - 1 = \pm 3i$$

$$z = 1 \pm 3i$$

$$f(z) = 0 \ \Rightarrow\ z = \frac{1}{2},\ 1+3i,\ 1-3i$$

Using synthetic

0.5	2	−5	22	−10
		1	−2	10
	2	−4	20	0

$$z = \frac{2 \pm \sqrt{4 - 4(10)}}{2}$$
$$= \frac{2 \pm \sqrt{-36}}{2}$$
$$= \frac{2 \pm 6i}{2} = 1 \pm 3i$$

alternate 2nd mark

Question 3 **[3 marks]**

A quadratic equation with real coefficients has one of its roots as $z = 7 - 2i$. Find the equation.

$$z = 7 \pm 2i \text{ are the roots}$$
$$\begin{aligned}(z - (7 + 2i))(z - (7 - 2i)) &= 0\\ (z - 7 - 2i)(z - 7 + 2i) &= 0\\ (z-7)^2 + 4 &= 0\\ z^2 - 14z + 49 + 4 &= 0\\ z^2 - 14z + 53 &= 0\end{aligned}$$

Question 4 **[3 marks]**

If $z = \frac{1}{2-3i}$ then express z in cartesian form. Hence find $z.\bar{z}$

$$\begin{aligned} Z &= \frac{1}{2-3i}\\ &= \frac{1}{2-3i} \times \frac{2+3i}{2+3i}\\ &= \frac{2+3i}{13}\\ Z &= \frac{2}{13} + \frac{3}{13}i\\ Z\bar{Z} &= \left(\frac{2}{13} + \frac{3}{13}i\right)\left(\frac{2}{13} - \frac{3}{13}i\right)\\ &= \frac{4}{169} - \frac{9}{169}i^2\\ &= \frac{13}{169}\\ Z\bar{Z} &= \frac{1}{13}\end{aligned}$$

$$\begin{aligned}|z| &= \sqrt{\left(\frac{2}{13}\right)^2 + \left(\frac{3}{13}\right)^2}\\ &= \sqrt{\frac{4}{169} + \frac{9}{169}}\\ &= \sqrt{\frac{13}{169}}\\ &= \sqrt{\frac{1}{13}}\\ &= \frac{1}{\sqrt{13}}\end{aligned}$$

alternate 2nd mark

$$z.\bar{z} = |z|^2 = \frac{1}{13}$$

Question 5 **[5 marks]**

Using de Moivre's Theorem, find the exact value of $(1+i)^5 - (1-i)^5$

$(\sqrt{2}\ \text{cis}\ 45^\circ)^5 - (\sqrt{2}\ \text{cis}\ (-45^\circ))^5$ ✓

$= 2^{\frac{5}{2}}\ \text{cis}\ 225^\circ - 2^{\frac{5}{2}}\ \text{cis}\ (-225^\circ)$

$= 2^{\frac{5}{2}}\ (\text{cis}\ 225^\circ - \text{cis}\ (-225^\circ))$ ✓

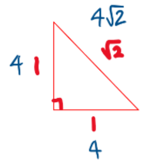
$= 2^{\frac{5}{2}}(\cos 225^\circ + i \sin 225^\circ$
$- (\cos (-225^\circ) + i \sin (-225^\circ))$ ✓

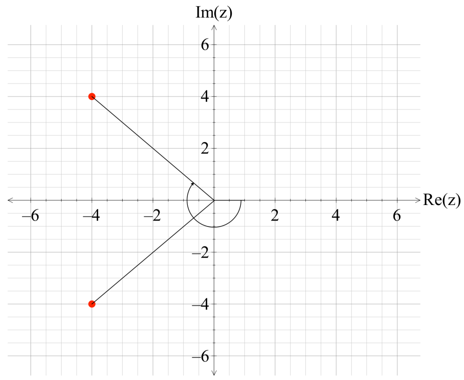
$= 2^{\frac{5}{2}}(\cos 225^\circ + i \sin 225^\circ$
$- \cos 225^\circ + i \sin 225^\circ)$ ✓

$= 2^{\frac{5}{2}}(2\ i \sin 225^\circ)$

$= 2^{\frac{7}{2}} \cdot i\left(-\frac{1}{\sqrt{2}}\right)$ ✓

$= -8i$





$= 4\sqrt{2}cis\ 225^\circ - 4\sqrt{2}cis\ (-225^\circ)$ ✓ ✓

$= -4 + 4i - (-4 - 4i)$ ✓

$= -8i$ ✓

alternate
2nd, 3rd, 4th
& 5th

Question 6 **[2 marks]**

Complete this proof that de Moivre's Theorem hold for $n = -1$. *Ensure you provide your reasons when you use various rules (or show use of them).*

RTP: $(cis\ \theta)^{-1} = cis\ (-\theta)$

Proof: LHS = $(cis\ \theta)^{-1}$

$= \frac{1}{\cos\theta + i\sin\theta} \times \frac{\cos\theta - i\sin\theta}{\cos\theta - i\sin\theta}$

$= \frac{\cos\theta - i\sin\theta}{\cos^2\theta - i^2\sin^2\theta}$ ✓

$= \frac{\cos(-\theta) - i(-\sin(-\theta))}{\cos^2\theta + \sin^2\theta}$

$= \frac{\cos(-\theta) + i\sin(-\theta)}{1}$ ✓

$= cis\ (-\theta)$

= RHS

QED

Year 12 Mathematics Specialist 2019

Test Number 1: Complex Numbers

Resource Rich

Name: ____ **Teacher: Mrs Da Cruz**

Marks: **25**

Time Allowed: **25 minutes**

Instructions: You are permitted 1 A4 page of notes and calculators. Show your working where appropriate remembering you must show working for questions worth more than 2 marks.

Question 7 **[1, 1, 2, 2 = 6 marks]**

Let two complex numbers be $z = a(\cos\theta + i\sin\theta)$ and $w = b(\cos\phi + i\sin\phi)$. Determine the following in terms of a, b, ϕ and θ.

a) $Arg(\sqrt{w})$.

$$\frac{\phi}{2}$$ ✓

b) $\frac{|w|}{|z^2|}$.

$$\frac{|w|}{|z^2|} = \frac{b}{a^2}$$ ✓

c) $Arg\left(\frac{zw}{i}\right)$.

$$i = cis\frac{\pi}{2}$$

✓ ✓

$$Arg\left(\frac{zw}{i}\right) = \theta + \phi - \frac{\pi}{2}$$

d) $|z^{-1}w^2|$.

$$\begin{aligned} |z^{-1}w^2| &= |z^{-1}| \times |w^2| \\ &= \frac{1}{a} \times b^2 \\ &= \frac{b^2}{a} \end{aligned}$$ ✓ ✓

Question 8 **[1, 1, 2 = 4 marks]**

The complex numbers z and $z + w$ are shown on the Argand diagram below. On the same diagram plot and label the location of the following.

a) $i^2 z$. $i^2 z = -z$ $z = 2+2i$

b) w. $z+w = -1+i$

$\therefore w = -3-i$

c) $z^2 + wz$.

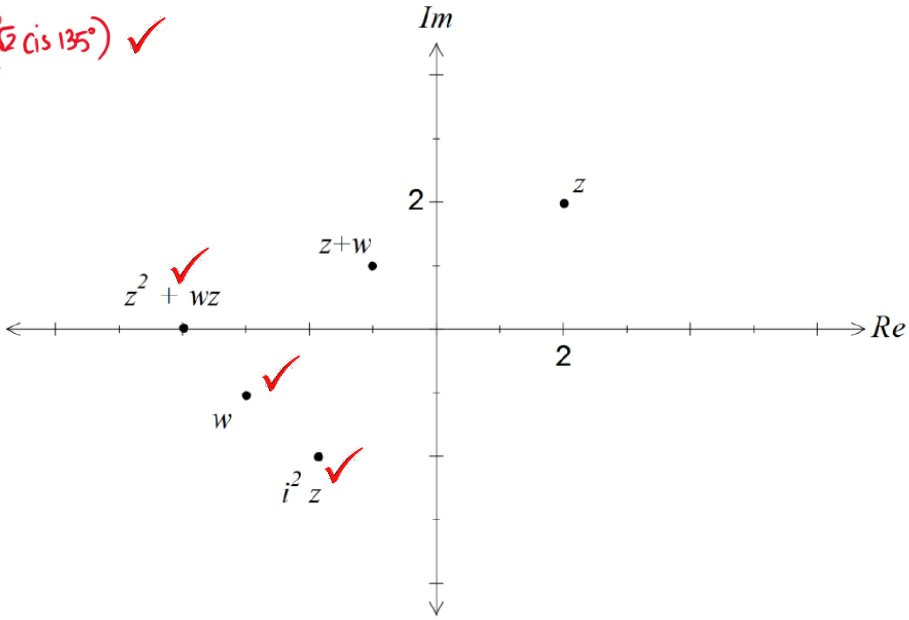
```
cExpand((2+2i)^2
                                 8·i
cExpand((2+2i)(−3−i)
                              −4−8·i
```

$= z(z+w)$

$= (2\sqrt{2}\text{ cis }45^\circ)(\sqrt{2}\text{ cis }135^\circ)$ ✓

$= 4\text{ cis }180^\circ$

$= -4$



Question 9 **[4 marks]**

Sketch the region $|z| = 2|z - 3|$ in the complex plane.

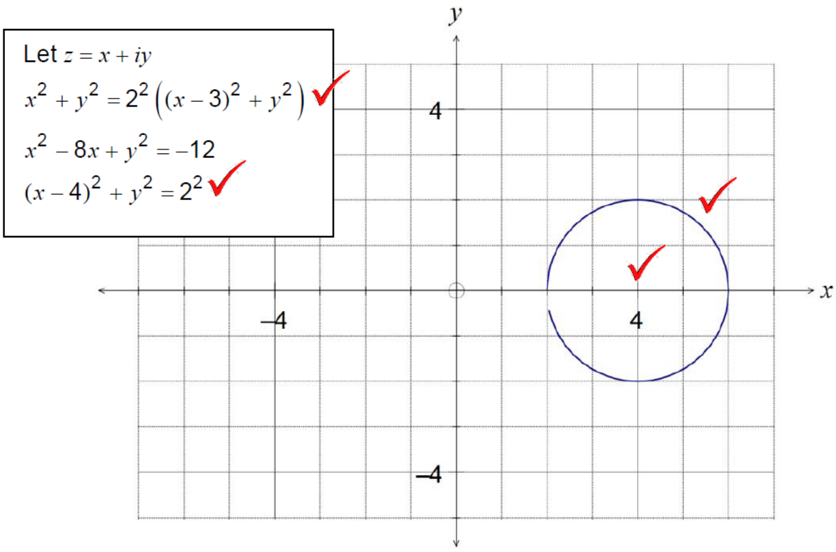
Must show working out for more than 2 marks.

Let $z = x + iy$

$x^2 + y^2 = 2^2\left((x-3)^2 + y^2\right)$ ✓

$x^2 - 8x + y^2 = -12$

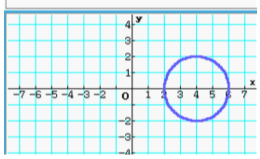
$(x-4)^2 + y^2 = 2^2$ ✓




```
|z|=2|z−3|
                   |x+y·i|=2·|x+y·i−3|
cExpand(ans)^2
                   x²+y²=4·((x−3)²+y²)
expand(
                   x²+y²=4·x²+4·y²−24·x+36
rewrite(
                   −3·x²−3·y²+24·x−36=0
solve(ans, y
        {y=−√(−x²+8·x−12), y=√(−x²+8·x−12)}
```

Conics Equation:

$(x-4)^2+y^2=4$



Question 10 **[4 marks]**

Solve the equation $z^4 = 2 + 2\sqrt{3}i$, expressing all solutions in polar form.

$$z^4 = 2^2 cis(\tfrac{\pi}{3})$$

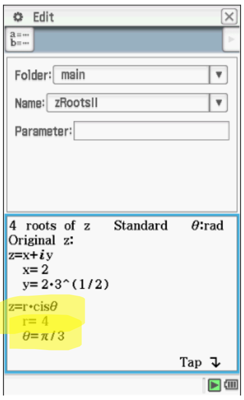
$$z_1 = \left(2^2\right)^{\frac{1}{4}} cis\left(\frac{\pi}{3} \times \frac{1}{4}\right)$$

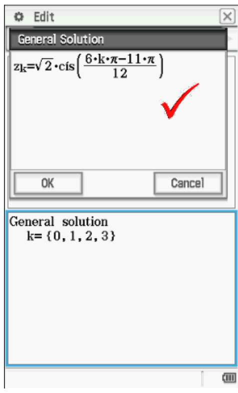
$$z_1 = \sqrt{2}cis(\tfrac{\pi}{12})$$

$$z_2 = \sqrt{2}cis(\tfrac{7\pi}{12})$$

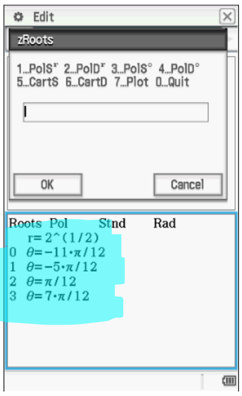
$$z_3 = \sqrt{2}cis(-\tfrac{5\pi}{12})$$

$$z_4 = \sqrt{2}cis(-\tfrac{11\pi}{12})$$





alternate 2^{nd} mark

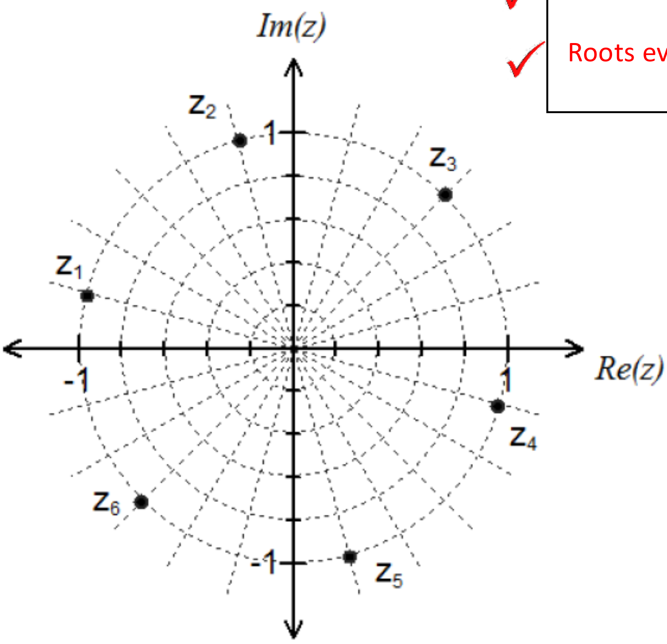


Question 11 **[2, 2 = 4 marks]**

One solution to the equation $z^6 = a + bi$, where a and b are real constants, is shown on the diagram below.

5 more roots all with modulus 1

Roots evenly spaced around the circle



(i) Plot all other solutions to the equation on the diagram. (2 marks)

(ii) Determine the values of a and b. (2 marks)

Using $z_3 = cis(\frac{\pi}{4})$

$z^6 = cis(\frac{6\pi}{4})$

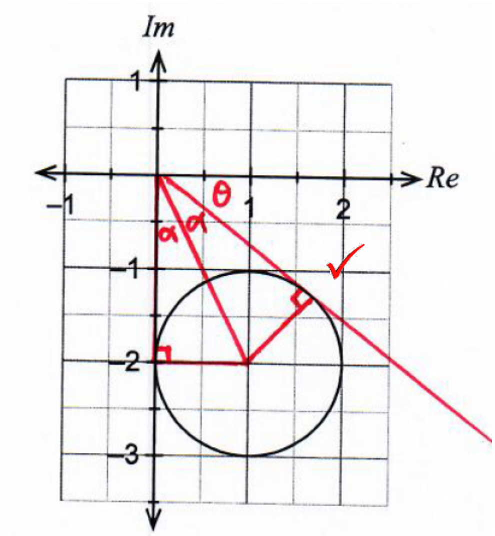
$= -i$

$a = 0, \ b = -1$

Note: students could use any of the 6 roots.

Question 12 **[3 marks]**

A sketch of the locus of a complex number z is shown below. Determine the maximum value for $\arg(z)$ correct to 0.01, where $0 \le \arg(z) < 2\pi$.



$\tan\alpha = \frac{1}{2}$

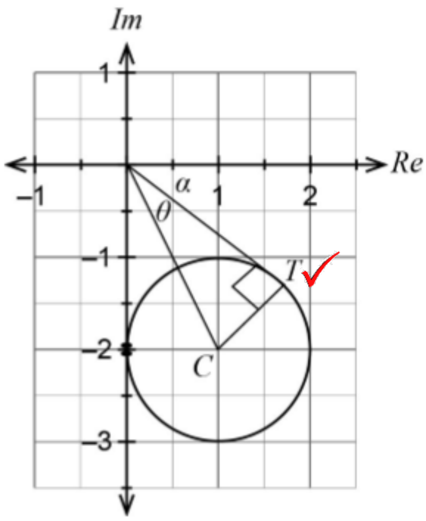
$\therefore \alpha = 26.565...^\circ = 0.4636...^r$

$\theta = \frac{\pi}{2} - 2\alpha$

$\therefore \theta = 0.64^r$

$\therefore Arg(z) = 5.64^r \quad (2\pi - \theta)$

Alternate presentation of solution | **Solution**



The tangent to the circle at point T gives the most positive value for $arg(z)$.

i.e. Maximum $arg(z) = 2\pi - \alpha$

In ΔOTC $\quad \sin\theta = \frac{1}{\sqrt{5}} \quad \therefore \quad \theta = 0.4636$

$\tan(\theta + \alpha) = 2 \quad \therefore \quad \theta + \alpha = 1.01071$

Hence $\alpha = 1.0107 - 0.4636 = 0.6435$

Hence the maximum value for $Arg(z) = 2\pi - 0.64 = 5.64$ (nearest 0.01).

Specific behaviours

✓ indicates how the maximum value occurs using the tangent
✓ uses appropriate trigonometry to determine θ correctly
✓ evaluates the maximum value for $arg(z)$ correctly